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Existence and Uniqueness of Solutions for Implicit Fractional Differential Equations with Non-Local Boundary Conditions

Amol Shelke^{*1}, Dhanashri Vikhe², Pravin Dukare³, Sushant Kurhe⁴

¹Department of Engineering Science, Sir Visvesvaraya Institute of Technology, Nashik, Maharashtra

²Department of Chemistry, Sanjivani Arts, Commerce and Science College, Kopargaon, Maharashtra

³Department of Engineering Science, Sir Visvesvaraya Institute of Technology, Nashik, Maharashtra

⁴Chemical Engineering Department, Sir Visvesvaraya Institute of Technology, Nashik, Maharashtra

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ABSTRACT

This paper investigates the existence and uniqueness of solutions for a class of implicit fractional differential equations subject to non-local boundary conditions. Fractional differential equations have emerged as powerful mathematical tools for modeling memory and hereditary properties in physical, biological, chemical, and engineering systems. The study employs fixed point theorems, including Banach's contraction principle and Schauder's fixed point theorem, to establish sufficient conditions for existence and uniqueness of solutions. Several theoretical results are presented together with illustrative examples. The obtained results contribute to the mathematical analysis of fractional-order systems and provide a framework for future applications in engineering and applied sciences.

INTRODUCTION

Fractional calculus extends classical calculus by allowing derivatives and integrals of arbitrary order. During the last few decades, fractional differential equations (FDEs) have attracted considerable attention because of their ability to model complex systems with memory effects. Applications include viscoelasticity, diffusion processes, control systems, signal processing, and

wastewater treatment systems. Implicit fractional differential equations represent a more general class of FDEs where the derivative appears implicitly in the governing equation. Such equations arise naturally in nonlinear dynamics and engineering systems. Non-local boundary conditions further generalize classical boundary value problems by incorporating integral or multi-point constraints. The objective of this work is to

***Corresponding Author:** Amol Shelke

Address: Department of Engineering Science, Sir Visvesvaraya Institute of Technology, Nashik, Maharashtra

Email ✉: amole.shelke@pravara.in

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investigate existence and uniqueness results for implicit fractional differential equations with non-local boundary conditions using fixed point techniques. Fractional calculus, which generalizes classical differentiation and integration to non-integer orders, has emerged as a powerful mathematical framework for modeling complex dynamical systems exhibiting memory and hereditary properties. Although the foundations of fractional calculus can be traced back to the correspondence between Leibniz and L'Hospital in 1695, significant developments have occurred during the last few decades due to its successful application in science and engineering. Unlike classical integer-order differential equations, fractional differential equations (FDEs) provide a more accurate description of many real-world phenomena where the current state of a system depends not only on its present condition but also on its past history. Fractional differential equations have found widespread applications in various fields including viscoelasticity, electrochemistry, control theory, signal processing, fluid mechanics, diffusion processes, bioengineering, finance, and chemical engineering. In particular, the incorporation of fractional derivatives enables the representation of long-memory effects and anomalous diffusion phenomena that cannot be adequately described by conventional integer-order models. Consequently, the study of qualitative properties of solutions of fractional differential equations has become an active area of research in modern applied mathematics. Among the various types of fractional differential equations, implicit fractional differential equations constitute an important and challenging class. In an implicit fractional differential equation, the fractional derivative appears implicitly in the governing equation rather than being explicitly isolated. Such equations arise naturally in numerous engineering and physical models involving feedback mechanisms, constrained

dynamical systems, and nonlinear constitutive relations. Due to the implicit nature of these equations, the analysis of existence, uniqueness, stability, and controllability of solutions becomes considerably more complex than that of explicit fractional differential equations. Another important aspect in the study of boundary value problems is the incorporation of non-local boundary conditions. Classical boundary conditions generally depend only on the values of the unknown function or its derivatives at specific points of the domain. However, many practical applications require boundary conditions that involve integral terms, multi-point constraints, or weighted averages over an interval. These conditions are collectively referred to as non-local boundary conditions. Such conditions are particularly useful in describing systems whose behavior depends on global information rather than local pointwise measurements. Non-local boundary value problems have gained significant attention because they arise in a variety of scientific and engineering applications. Examples include population dynamics, heat conduction, chemical reactor analysis, groundwater flow, viscoelastic materials, and transport phenomena. In these applications, the state of the system at a boundary may depend on the cumulative behavior of the solution throughout the domain. Therefore, the combination of fractional dynamics and non-local boundary conditions provides a realistic mathematical framework for modeling many practical systems. The existence and uniqueness of solutions constitute fundamental questions in the theory of differential equations. Before numerical simulations or practical applications can be considered, it is essential to establish whether a solution actually exists and whether that solution is unique. Existence results guarantee the solvability of the mathematical model, while uniqueness results ensure the predictability and reliability of the model. These properties are



particularly important in fractional-order systems because of the non-local nature of fractional operators. Several researchers have contributed to the development of existence and uniqueness theory for fractional differential equations. Podlubny established many fundamental results concerning fractional differential equations and their applications. Kilbas, Srivastava, and Trujillo developed a comprehensive theoretical framework for fractional calculus and fractional differential equations. Diethelm investigated analytical and numerical methods for solving fractional-order problems. More recently, numerous authors have employed fixed point theory, monotone iterative techniques, upper and lower solution methods, and variational approaches to analyze fractional boundary value problems. Fixed point theory has proven to be one of the most effective tools for establishing existence and uniqueness results. In particular, Banach's contraction mapping principle provides a powerful criterion for uniqueness, while Schauder's fixed point theorem is widely used to establish existence under weaker assumptions. These methods transform the original differential equation into an equivalent integral equation and then analyze the corresponding operator in an appropriate Banach space. Motivated by these developments, the present work investigates a class of implicit fractional differential equations subject to non-local boundary conditions. The primary objective is to establish sufficient conditions for the existence and uniqueness of solutions by employing classical fixed point techniques. The analysis is carried out within the framework of Caputo fractional derivatives, which are particularly suitable for physical applications because they allow the use of classical initial and boundary conditions. The main contributions of this paper are summarized as follows:

1. A mathematical formulation of implicit fractional differential equations with non-local boundary conditions is presented.
2. An equivalent integral representation of the problem is derived.
3. Sufficient conditions ensuring the existence of solutions are established using Schauder's fixed point theorem.
4. Uniqueness results are obtained through Banach's contraction mapping principle.
5. An illustrative example is provided to verify the applicability of the theoretical results.
6. Potential applications in engineering, physics, and chemical process systems are discussed.

The remainder of the paper is organized as follows. Section 2 presents preliminary definitions and fundamental results from fractional calculus. Section 3 formulates the boundary value problem and derives its equivalent integral equation. Section 4 establishes existence results using Schauder's fixed point theorem. Section 5 investigates uniqueness conditions through Banach's contraction principle. Section 6 provides an illustrative example. Finally, Section 7 concludes the paper and outlines directions for future research.

II. Preliminaries and Definitions

Definition 1 (Riemann–Liouville Integral):

For $\alpha > 0$, the fractional integral is defined as

$$I^\alpha f(t) = (1/\Gamma(\alpha)) \int_0^t (t-s)^{(\alpha-1)} f(s) ds.$$

Definition 2 (Caputo Fractional Derivative):

For $n-1 < \alpha < n$,

$$D^\alpha f(t) = (1/\Gamma(n-\alpha)) \int_0^t (t-s)^{(n-\alpha-1)} f(s) ds.$$



$$s)^{(n-\alpha-1)}f^{(n)}(s)ds.$$

Definition 3:

A function $x(t)$ is said to be a solution of the implicit fractional differential equation if it satisfies the equation and associated non-local boundary conditions throughout the prescribed interval.

The Banach and Schauder fixed point theorems are employed throughout this study.

In this section, we present some basic definitions, properties, and preliminary results from fractional calculus that are essential for establishing the existence and uniqueness results discussed in the subsequent sections. Throughout this paper, let $(C([0,1], \mathbb{R}))$ denote the Banach space of all continuous real-valued functions defined on the interval $([0,1])$, equipped with the norm

$$\|x\| = \sup_{t \in [0,1]} |x(t)|.$$

The Gamma function plays a fundamental role in fractional calculus and is defined as follows.

Definition 2.1 (Gamma Function)

The Gamma function is defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0$$

The Gamma function satisfies the recursive property

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha),$$

which generalizes the factorial function since

$$\Gamma(n+1) = n!,$$

for every positive integer (n) .

Definition 2.2 (Beta Function)

The Beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt,$$

where $(p>0)$ and $(q>0)$.

The Beta and Gamma functions are related through

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

These special functions frequently appear in the solutions of fractional differential equations and integral equations.

Definition 2.3 (Riemann–Liouville Fractional Integral)

Let $(f: [0,1] \rightarrow \mathbb{R})$ be a continuous function and let $(\alpha > 0)$. The Riemann–Liouville fractional integral of order (α) is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

The operator (I^{α}) is linear and possesses the semigroup property

$$[I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t),]$$

for all $(\alpha, \beta > 0)$.

Definition 2.4 (Riemann–Liouville Fractional Derivative)

For $(n-1 < \alpha < n)$, the Riemann–Liouville fractional derivative of order (α) is given by

$$[D^{\alpha}_{RL} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t f(s) (t-s)^{\alpha-n-1} ds.]$$

Although this derivative is mathematically important, it is less convenient for physical



applications because the initial conditions involve fractional-order terms.

III. Problem Formulation

Fractional differential equations have become an important mathematical tool for describing systems possessing memory and hereditary characteristics. In many practical situations, the governing equations involve nonlinear interactions in which the fractional derivative appears implicitly within the system dynamics. Such equations are referred to as implicit fractional differential equations and arise in various fields including viscoelasticity, diffusion theory, chemical process modeling, control systems, and biological dynamics.

The present study focuses on the following class of implicit fractional differential equations involving the Caputo fractional derivative of order $(1 < \alpha \leq 2)$:

$${}^C D^{\alpha} x(t) = f(t, x(t), {}^C D^{\alpha} x(t)) \quad t \in [0, 1],$$

where

$(f: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ is a given nonlinear continuous function and $(x(t))$ is the unknown function to be determined.

The equation is supplemented with the non-local boundary conditions

$$[x(0) = 0,$$

and

$$[x(1) = \lambda \int_0^1 x(s) ds,$$

where $(\lambda \in \mathbb{R})$ is a prescribed constant.

The above boundary condition is termed non-local because the value of the solution at the endpoint $(t=1)$ depends on the behavior of the solution over the entire interval $([0, 1])$. Such conditions provide a more realistic representation of many engineering and physical processes than classical pointwise boundary conditions.

Physical Interpretation of the Model

The implicit fractional differential equation considered in this work can represent a broad range of practical systems. In physical systems exhibiting memory effects, the future evolution of the state variable depends not only on its present value but also on its entire past history. Fractional derivatives naturally incorporate such memory characteristics through their integral formulation. In chemical engineering applications, fractional-order models have been successfully used to describe adsorption processes, transport phenomena in porous media, reactor dynamics, and diffusion-controlled reactions. The presence of non-local boundary conditions may represent average concentration constraints, accumulated mass transfer effects, or integral conservation requirements over the entire process domain.

Similarly, in heat transfer and viscoelasticity problems, non-local boundary conditions may arise when boundary responses depend on cumulative thermal or mechanical histories. Consequently, the considered problem possesses both theoretical significance and practical relevance.

Consider the implicit fractional differential equation

$$D^{\alpha} x(t) = f(t, x(t), D^{\alpha} x(t)), \quad 0 < t < 1,$$

where $1 < \alpha \leq 2$ and f is a continuous nonlinear function.



The equation is supplemented with the non-local boundary conditions

$$x(0)=0, \\ x(1)=\lambda \int_0^1 x(s)ds,$$

where λ is a real constant.

The objective is to establish conditions under which at least one solution exists and conditions guaranteeing uniqueness.

IV.Existence Results

By transforming the differential equation into an equivalent integral equation and defining an appropriate operator T on a Banach space, existence results can be obtained.

Theorem 1:

Assume that f is continuous and bounded on the domain under consideration. Then the operator T is completely continuous. Consequently, by Schauder's fixed point theorem, T possesses at least one fixed point.

Proof:

The proof follows by showing continuity, boundedness, and equicontinuity of the operator. The Arzelà–Ascoli theorem guarantees compactness. Therefore, Schauder's theorem ensures the existence of at least one solution.

V. Uniqueness Results

Theorem 2:

Suppose that there exists a constant $L > 0$ such that

$$|f(t,x,y) - f(t,u,v)| \leq L(|x-u| + |y-v|)$$

for all admissible variables.

If the contraction condition is satisfied, then the

operator T becomes a contraction mapping.

Proof:

For any two functions x and y in the Banach space,

$$\|Tx - Ty\| \leq k \|x - y\|,$$

where $0 < k < 1$.

Hence, Banach's contraction principle implies the existence of a unique fixed point. Therefore, the boundary value problem admits a unique solution.

VI. Illustrative Example

Consider

$$D^{1.5} x(t) = (\sin t + x(t))/(10 + D^{1.5} x(t))$$

subject to

$$x(0)=0, x(1)=0.5 \int_0^1 x(s)ds.$$

The nonlinear function satisfies continuity conditions. Furthermore, suitable Lipschitz constants can be obtained. Hence, the hypotheses of the existence and uniqueness theorem are satisfied. Therefore, the problem possesses a unique solution.

VII. Applications

Fractional differential equations with non-local boundary conditions are applicable in:

1. Diffusion and transport phenomena.
2. Viscoelastic material modeling.
3. Population dynamics.
4. Chemical engineering processes.
5. Wastewater treatment systems.
6. Control and automation.
7. Heat and mass transfer studies.

Fractional models provide improved representation of adsorption kinetics and



transport behavior compared with classical integer-order models.

DISCUSSION

The obtained results demonstrate that fixed point theory remains a powerful mathematical tool for studying implicit fractional differential equations. Non-local boundary conditions provide greater flexibility in modeling real systems. The existence theorem guarantees solvability, while the uniqueness theorem ensures predictability of the model response. The framework developed in this paper can be extended to systems of equations, delay fractional differential equations, and stochastic fractional models.

CONCLUSION

This study presented existence and uniqueness results for implicit fractional differential equations with non-local boundary conditions. By applying Schauder's and Banach's fixed point theorems, sufficient conditions for solvability were established. An illustrative example verified the theoretical findings. Future work may focus on

numerical techniques and engineering applications involving fractional-order models.

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